06-07-2025

Mathematical Building Blocks

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DCS, CUSAT

EQUATION OF A LINE

slope → $y = mx + b$ ← constant

variables

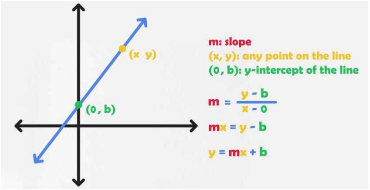
$\Longleftrightarrow$

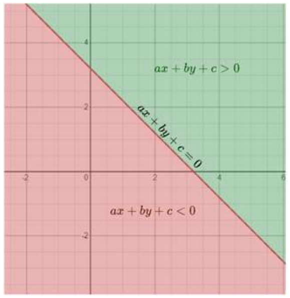
$Ax + By = C$,

where :

A, B, C = constants (only integers!)

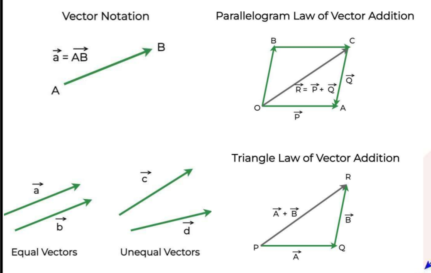
x, y = variables

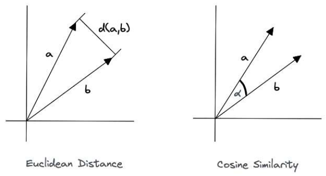


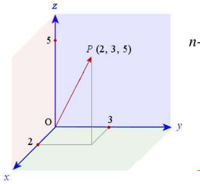


VECTORS AND VECTORSPACE

A vector is a quantity or phenomenon that has two independent properties: magnitude and direction.





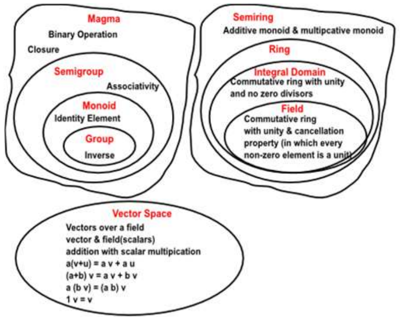


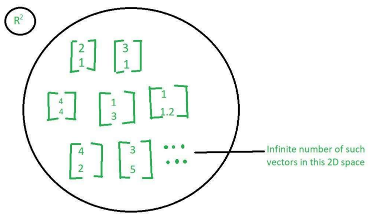
n-Dimensional Vectors and Points

$$v = (v_1, v_2, \cdots, v_n)$$

$$v' = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

VECTORS AND VECTORSPACE

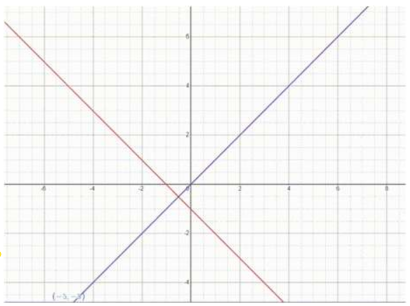




$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Basis of R^2

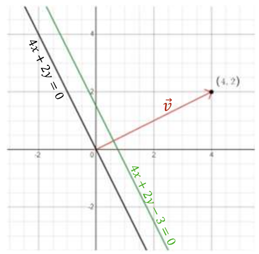
LINE AND NORMAL



The line $ax + by + c = 0$

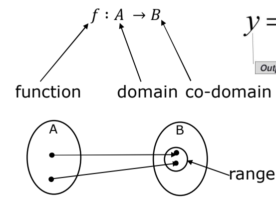
and the vector (a, b)

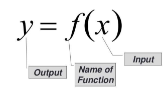
$\vec{v} = (a, b)$ *is always normal to* $ax + by + c = 0$

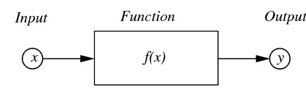


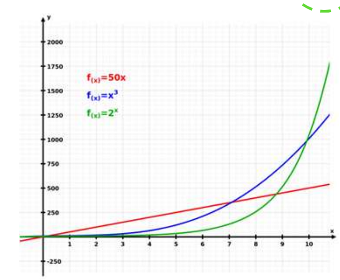
FUNCTIONS

A function relates every element in a set to exactly one element in another set



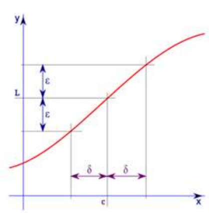






A function which has either or one of its subsets as its range is called a **real valued function**. Further, if its domain is also either R or a subset of R, it is called a **real function**

LIMIT OF A FUNCTION



$$\lim_{\underbrace{x\to a}_{\text{"As you approach } a \text{ along the x-axis"}}} \overbrace{f(x)}^{\text{function}} = \underbrace{L}_{\text{"What is the y-value getting closer to?"}}$$

The **limit** of a function at a point *a* in its domain (if it exists) is the value that the function approaches as its argument approaches *a*

$\lim_{x\to c^-} f(x) = \lim_{h\to 0} f(x-h)$

Left Hand Limit

$\lim_{x\to c^+} f(x) = \lim_{h\to 0} f(x+h)$

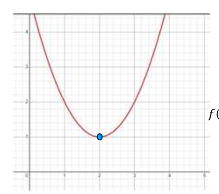
Right Hand Limit

The limit of a function exists if and only if the left-hand limit is equal to the right-hand limit.

$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$

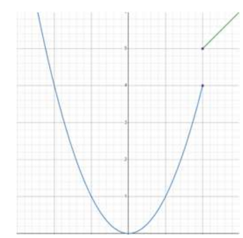
$$\lim_{x\to 5} f(x) = x + 4 = 9$$

LIMIT OF A FUNCTION



$f(x) = (x-2)^2 + 1$

x^-	$f(x^-)$		x^+	$f(x^+)$
1	2		3	2
1.5	1.25		2.5	1.25
1.9	1.01		2.1	1.01
1.99	1.0001		2.01	1.0001
1.999	1.000001		2.001	1.000001
1.9999	1.00000001		2.0001	1.00000001



$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x + 3 & \text{otherwise} \end{cases}$

x^-	$f(x^-)$		x^+	$f(x^+)$
1	1		3	6
1.5	2.25		2.5	5.5
1.9	3.61		2.1	5.1
1.99	3.9601		2.01	5.01
1.999	3.996001		2.001	5.001
1.9999	3.99960001		2.0001	5.0001

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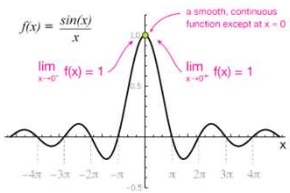
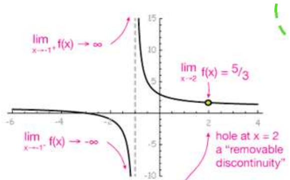
LIMIT OF A FUNCTION



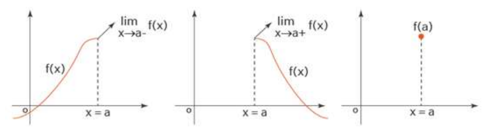


$$f(x) = \frac{x^2+x-6}{x^2-x-2} = \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}(x+1)}$$

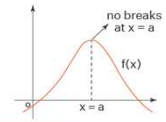
x	sin(x)/x	x	sin(x)/x
π	0.0	-π	0.0
1.0000	0.8414710	-1.0000	0.8414710
0.1000	0.9983342	-0.1000	0.9983342
0.0010	0.9999998	-0.0010	0.9999998
0.0001	1.0000000	-0.0001	1.0000000

$x\to2^-$	f(x)	$x\to2^+$	f(x)
1.50000	1.80000	2.50000	1.57143
1.75000	1.72727	2.25000	1.61538
1.90000	1.68966	2.10000	1.64516
1.95000	1.67797	2.05000	1.65574
1.99000	1.66890	2.01000	1.66445
1.99900	1.66689	2.00100	1.66644
1.99990	1.66669	2.00010	1.66664
1.99999	1.66667	2.00001	1.66666

CONTINUITY



These three together will make the function f(x) continous at x = a



$\therefore \boxed{\lim_{x\to a} f(x) = f(a)} \Rightarrow$ f(x) is continuous at x = a

Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

$f(x)$ is continuous at $x = 1$

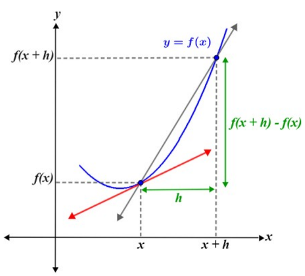
if $\lim_{x\to1} f(x) = f(1)$

L.H.S	R.H.S
$\lim_{x\to1} f(x)$	$f(1)$
$= \lim_{x\to1} (2x+3)$	$= 2\times1+3$
$= 2\times1+3$	$= 2+3$
$= 2+3$	**= 5**
= 5	

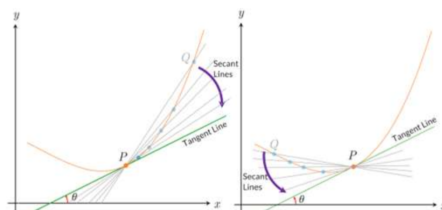
Since, L.H.S = R.H.S

∴ Function is **continuous.**

DERIVATIVE OF A FUNCTION



$$f'(x) = \frac{df}{dx} = \lim_{\Delta x\to0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to0} \frac{\Delta f}{\Delta x}$$



Slope of Secant = $\frac{f(x+h) - f(x)}{h}$

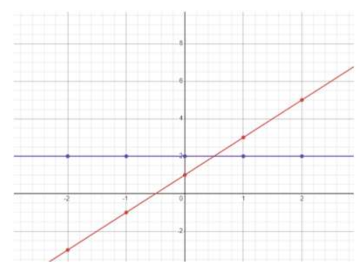
("Difference quotient")

$f'(x) = \lim_{h\to0} \frac{f(x+h) - f(x)}{h}$

(if limit exists)

DERIVATIVE OF A FUNCTION

$f(x) = 2x + 1$ $\quad \frac{d}{dx} f(x) = 2$



x	1	2	3	4	5	6	7	8
f(x)	3	5	7	9	11	13	15	17

$\frac{d}{dx} f(x) = \frac{5-3}{2-1} = \frac{2}{1} = 2$

$\frac{d}{dx} f(x) = \frac{9-3}{4-1} = \frac{6}{3} = 2$

$\frac{d}{dx} f(x) = \frac{7-5}{3-2} = \frac{2}{1} = 2$

$\frac{d}{dx} f(x) = \frac{17-9}{8-4} = \frac{8}{4} = 2$

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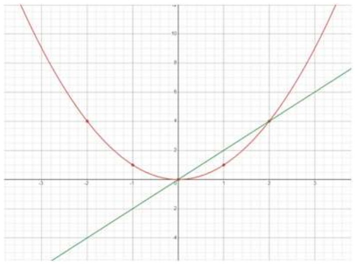
DERIVATIVE OF A FUNCTION

$f(x) = x^2$ $\qquad$ $\frac{d}{dx}f(x) = 2x$



x	0	1	2	3	4	5	6	7
$f(x)$	0	1	4	9	16	25	36	49

1 $\quad$ 3 $\quad$ 5 $\quad$ 7 $\quad$ 9

$\frac{(1+3)}{2} = \mathbf{2}$ $\quad$ $\frac{(3+5)}{2} = \mathbf{4}$ $\quad$ $\frac{(5+7)}{2} = \mathbf{6}$ $\quad$ $\frac{(7+9)}{2} = \mathbf{8}$

$\frac{d}{dx}f(x)$ *at* $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4

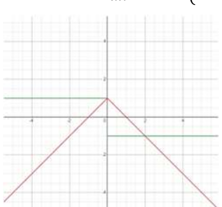
so the derivative of $\boldsymbol{f(x) = x^2}$ $\qquad$ $\boldsymbol{\frac{d}{dx}f(x) = 2x}$

$f(x) = x^2$ $\qquad$ $\frac{d}{dx}f(x) = 2x$

	x	f(x)	$\frac{df}{dx} = \frac{f(x_0)-f(x)}{x_0 - x}$			x	f(x)	$\frac{df}{dx} = \frac{f(x_0)-f(x)}{x_0 - x}$			x	f(x)	$\frac{df}{dx} = \frac{f(x_0)-f(x)}{x_0 - x}$
x_0^-	**0**	**0**	**1**			**1**	**1**	**3**			**2**	**4**	**5**
	0.1	0.01	1.1			1.1	1.21	3.1			2.1	4.41	5.1
	0.5	0.25	1.5			1.5	2.25	3.5			2.5	6.25	5.5
	0.9	0.81	1.9			1.9	3.61	3.9			2.9	8.41	5.9
	0.99	0.9801	1.99			1.99	3.9601	3.99			2.99	8.9401	5.99
	0.999	0.998001	1.999			1.999	3.996001	3.999			2.999	8.994001	5.999
x_0	**1**	**1**	**2**			**2**	**4**	**4**			**3**	**9**	**6**
	1.001	1.002001	2.001			2.001	4.004001	4.001			3.001	9.006001	6.001
	1.01	1.0201	2.01			2.01	4.0401	4.01			3.01	9.0601	6.01
	1.1	1.21	2.1			2.1	4.41	4.1			3.1	9.61	6.1
	1.5	2.25	2.5			2.5	6.25	4.5			3.5	12.25	6.5
	1.9	3.61	2.9			2.9	8.41	4.9			3.9	15.21	6.9
x_0^+	**2**	**4**	**3**			**3**	**9**	**5**			**4**	**16**	**7**

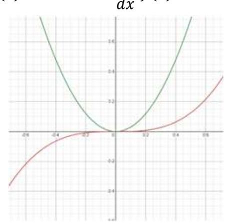
WHY DERIVATIVE EXSIST AND NOT EXSIST ?

$f(x) = 1 - |x|$ $\qquad$ $\frac{d}{dx}f(x) = \begin{cases} -1 & if\ x > 0 \\ 1 & if\ x < 0 \end{cases}$



h	$\frac{f(x-h)-f(x)}{-h}$	$\frac{f(x+h)-f(x)}{h}$
1	1	-1
0.1	1	-1
0.01	1	-1
0.001	1	-1
0.0001	1	-1
0.00001	1	-1
0.000001	1	-1

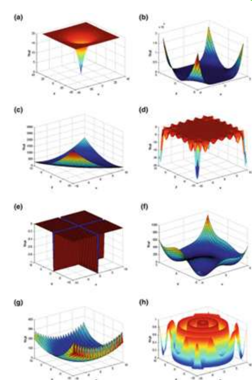
$f(x) = x^3$ $\qquad$ $\frac{d}{dx}f(x) = 2x^2$



h	$\frac{f(x-h)-f(x)}{-h}$	$\frac{f(x+h)-f(x)}{h}$
1	1	1
0.1	0.1	0.1
0.01	0.01	0.01
0.001	0.001	0.001
0.0001	0.0001	0.0001
0.00001	0.00001	0.00001
0.000001	0.000001	0.000001

FUNCTION OF SEVERAL REAL VARIABLES

A **real-valued function of** ***n*** **real variables** is a function that takes as input n real numbers, commonly represented by the variables $x_1, x_2, \ldots, x_n$ for producing another real number, the *value* of the function, commonly denoted $f(x_1, x_2, \ldots, x_n)$.



$y = f(x_1, x_2)$

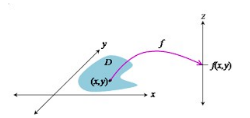
$y = f(x_1, x_2, x_3)$

.

.

.

$y = f(x_1, x_2, x_3, \ldots \ldots, x_n)$

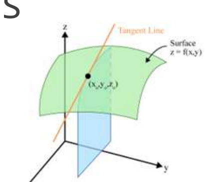


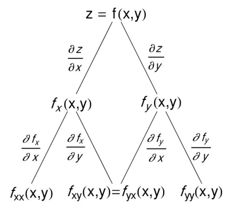
PARTIAL DERIVATIVES

Partial derivatives of a function of two variables states that if $z = f(x, y)$, then the first order partial derivatives of f with respect to x and y, provided the limits exist and are finite, are:

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

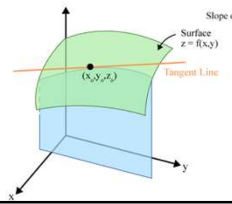




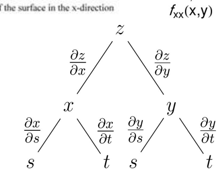
$z = xy^2 - y\sin(x)$

$\frac{\partial z}{\partial x} = y^2 - y\cos(x)$

$\frac{\partial y}{\partial x} = 2xy - \sin(x)$



Slope of the surface in the x-direction



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

PARTIAL DERIVATIVES ! WHY ?

Isolating Independent Effects: When a function depends on multiple variables, partial derivatives allow us to isolate the effect of changing one variable while holding the others constant. This is crucial for understanding how different factors interact and contribute to the overall behavior of the function

Local Sensitivity Analysis: Partial derivatives measure the instantaneous rate of change of a function with respect to each variable at a specific point. This provides insights into how sensitive the function is to small changes in each variable locally.

$z = f(x, y) = 2x + 3y$

$\frac{\partial z}{\partial x} = 2 \quad \frac{\partial z}{\partial y} = 3$

$f(1,0) = 2 \rightarrow f(3,4) = 18$

$\Delta f = 18 - 2 = 16$

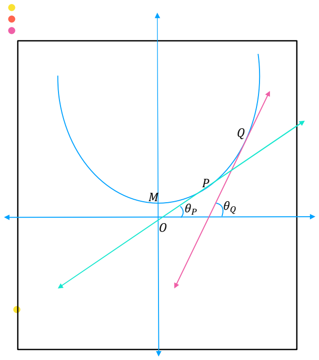
$\Delta x = 2 \quad \Delta y = 4$

$\Delta f = \Delta x\,.2 + \Delta y\,.3 = 16$

$\Delta f = \Delta x \frac{\partial z}{\partial x} + \Delta y \frac{\partial z}{\partial y} = 16$

f		y				
		0	1	2	3	4
x	0	0	3	6	9	12
	1	2	5	8	11	14
	2	4	7	10	13	16
	3	6	9	12	15	18
	4	8	11	14	17	20

DERIVATIVES AND OPTIMIZATION



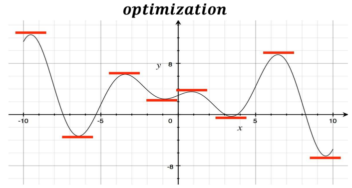
Here $\theta_p < \theta_Q \Leftrightarrow tan(\theta_P) < \tan(\theta_Q)$; we know,

$\frac{dy}{dx}$ at $P = tan(\theta_P)$; $\frac{dy}{dx}$ at $Q = tan(\theta_Q)$

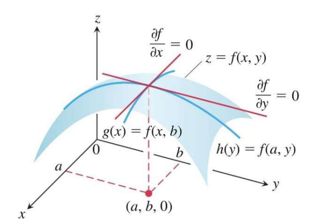
What is the angle of tangent at M ?

$\theta_m = 0 \Rightarrow \tan(\theta_m) = 0 \Rightarrow \frac{dy}{dx} = 0$ at M

Derivative at a point x_c is $0 \Rightarrow x_c$ is an extremum point which is a maximum or minimum, then it can be used for ***optimization***



DERIVATIVES AND OPTIMIZATION



Gradient of a function

$$\nabla f(x_1, x_2, x_3, \ldots, x_n) = \begin{bmatrix} \frac{\delta}{\delta x_1}(f) \\ \frac{\delta}{\delta x_2}(f) \\ \frac{\delta}{\delta x_3}(f) \\ \vdots \\ \frac{\delta}{\delta x_n}(f) \end{bmatrix}$$

Find the extrema of $f(x, y) = x^2 + 2y^2 - 4(x + y)$

1. Gradient:

$$\nabla f = (2x - 4, 4y - 4)$$

2. Critical point: (2, 1)

To find optimal points of a function $f(x_0, x_1, x_2, \ldots\ldots x_n)$:

Calculate ∇f

Find $x_0, x_1, \ldots\ldots x_n$ where $\nabla f(x_0, x_1, x_2, \ldots\ldots x_n) = 0$

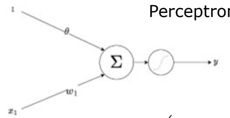
ERROR FUNCTION AND SURFACE

Linear Regression

$\hat{y} = mx + b$

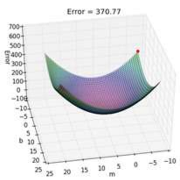
$J = \sum (y - \hat{y})^2 = \sum (y - mx - b)^2 = J(m, b)$

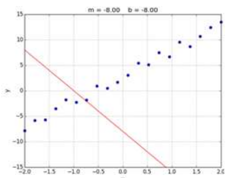
Perceptron

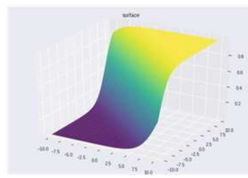


$\hat{y} = \frac{1}{1 + e^{(-w_1 x_1 - \theta}}$

$J = \sum (y - \hat{y})^2 = \sum \left(y - \frac{1}{1 + e^{(-w_1 x_1 - \theta)}}\right)^2 = J(w_1, \theta)$







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