Probability and Estimation Basics

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June 10, 2025

Basic Probability Terminologies

- **Sample Space (S):** Set of all possible outcomes.
- **Event (E):** Subset of the sample space.
- **Random Experiment:** Process that produces uncertain outcomes.
- **Trial:** A single performance of the experiment.
- **Mutually Exclusive Events:** Events that cannot occur together.
- **Exhaustive Events:** All possible outcomes are covered.

Basics of Probability

- Probability quantifies uncertainty.
- For an event E, the probability is:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total outcomes}}$$

- Range: $0 \leq P(E) \leq 1$

Conditional Probability

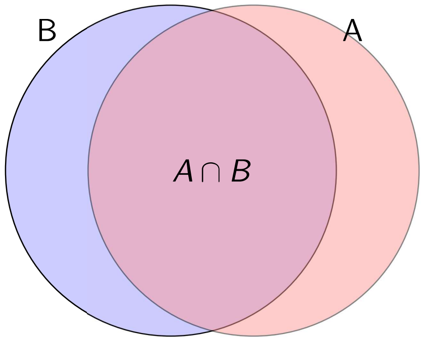
- Probability of event A given event B has occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

- **Example:**
  - Let $A =$ "Student passed Math", $B =$ "Student passed Physics"
  - If 30% passed both, and 50% passed Physics:

$$P(A|B) = \frac{0.3}{0.5} = 0.6$$

Conditional Probability Visualization



Sample space

Rules of Probability

- **Complement Rule:** $P(A^c) = 1 - P(A)$
- **Addition Rule:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **Multiplication Rule:**

$$P(A \cap B) = P(A)P(B|A)$$

Random Variables

- A random variable (RV) maps outcomes to real numbers.
- Notation: $X : S \to \mathbb{R}$
- **Discrete RV:** Countable values

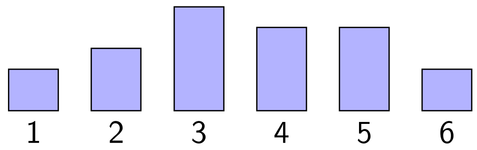
$$X = \text{Number on a die}, \quad P(X = x) = \frac{1}{6}, \ x \in \{1, 2, \ldots, 6\}$$

- **Continuous RV:** Values in intervals

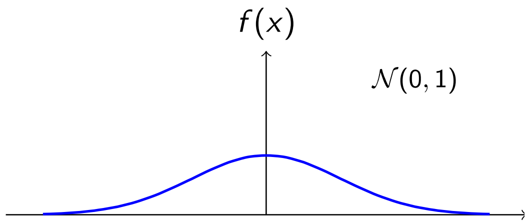
$$X = \text{Temperature in } {}^\circ\text{C}, \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

PMF and PDF Illustration

Discrete (PMF)



Continuous (PDF)



Bayes' Theorem – Limited Form

Bayes' Theorem:

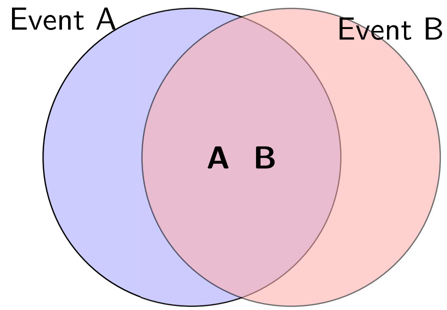
$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Where:

- $P(A|B)$ = Posterior probability (probability of event A given that B has occurred)
- $P(B|A)$ = Likelihood (probability of event B given that A is true)
- $P(A)$ = Prior probability of event A
- $P(B)$ = Total probability of event B

Note: This form is used when there are only two events involved, without a partition of the sample space.

Bayes Theorem Illustration



Overlap corresponds to
$P(A \cap B) = P(B)P(A|B)$

Bayes' Theorem (General Form)

- Let $\{B_1, B_2, \ldots, B_n\}$ be a partition of the sample space and A be any event.
- Then:

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}$$

- **Application:** Medical diagnosis, spam filtering, decision systems

Bayes' Theorem: Numerical Example

Problem:

- 1% of a population has a disease.
- The test is 99% accurate:
  - $P(\text{Positive}|\text{Disease}) = 0.99$
  - $P(\text{Positive}|\text{No Disease}) = 0.01$
- What is the probability that a person has the disease if they test positive?

Bayes' Theorem: Numerical Example

Solution:

- Let:

$$D = \text{has disease}, \quad \bar{D} = \text{no disease}, \quad T = \text{test positive}$$

- We want: $P(D|T) = \frac{P(T|D)P(D)}{P(T)}$
- Calculate:

$$P(T) = P(T|D)P(D) + P(T|\bar{D})P(\bar{D})$$

$$P(T) = (0.99)(0.01) + (0.01)(0.99) = 0.0198$$

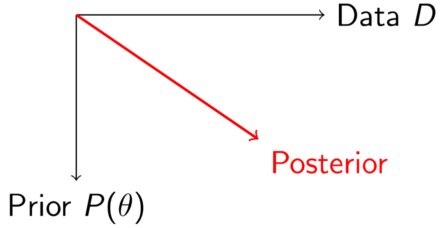
- Therefore:

$$P(D|T) = \frac{0.99 \times 0.01}{0.0198} = \frac{0.0099}{0.0198} = 0.5$$

Bayesian Inference View

- **Prior:** Initial belief about parameter θ: $P(\theta)$
- **Likelihood:** How likely data is under θ: $P(D|\theta)$
- **Posterior:** Updated belief:

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$



Summary

- Random variables model data: discrete for counts, continuous for measurements.
- Bayes' theorem updates belief in light of evidence.
- Generalization allows handling multiple hypotheses.
- Visualizations help interpret conditional dependencies.